PENCILS OF QUADRICS AND GROMOV-WITTEN-WELSCHINGER INVARIANTS OF $\mathbb{C}P^3$

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ABSTRACT. We establish a formula for the Gromov-Witten-Welschinger invariants of $\mathbb{C}P^3$ with mixed real and conjugate point constraints. The method is based on a suggestion by J. Kollár that, considering pencils of quadrics, some real and complex enumerative invariants of $\mathbb{C}P^3$ could be computed in terms of enumerative invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and of elliptic curves.

Invariant signed counts of real rational curves with point constraints in real surfaces and in many real threefolds were first defined by J.-Y. Welschinger in [Wel05a] and [Wel05b], respectively. In the case of surfaces, various methods for computation of these invariants are established [Mik05, Shu06, Wel07, BM08, HS12, IKS13, KR15, Bru15]. In the case of threefolds, methods for computation are developed only in the extremal cases: when all point constraints are real [BM07], when the number of real point constraints is minimal [Wel07, GZ13], and in the case of no constraints [PSW08]. In this paper, we establish a relation between the GWW invariants of $\mathbb{C}P^3$ and the GWW invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1$, which allows the computation of the former in terms of the latter. To our knowledge, Theorem 1 provides the first systematic explicit computation of the Welschinger invariants of $\mathbb{C}P^3$, and more generally of a real algebraic variety of dimension 3, when mixed real and conjugate point constraints are used.

Throughout the text, we equip $\mathbb{C}P^3$ and $\mathbb{C}P^1$ with their standard real structure defined by the complex conjugation. We equip $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the product of the standard real structure of $\mathbb{C}P^1$, in particular its real part is $\mathbb{R}P^1 \times \mathbb{R}P^1$.

Given $d \in \mathbb{Z}_{>0}$ and $\underline{x}$ a generic real configuration of $2d$ points in $\mathbb{C}P^3$, with $2l$ non-real points, we denote by $W_{\mathbb{R}P^3}(d, l)$ the corresponding Welschinger invariant counting with a sign the real rational curves of degree d in $\mathbb{C}P^3$ passing through all points in $\underline{x}$. Similarly, given $a, b \in \mathbb{Z}_{\geq 0}$ and $\underline{x}$ a generic real configuration of $2(a + b) - 1$ points in $\mathbb{C}P^1 \times \mathbb{C}P^1$, with $2l$ non-real points, we denote by $W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a, b), l)$ the Welschinger invariant counting with a sign the real rational curves of bidegree (a, b) in $\mathbb{C}P^1 \times \mathbb{C}P^1$ passing through all points in $\underline{x}$.

Theorem 1. *For every odd positive integer d and $l \in \{0, \ldots, d-1\}$, one has*

$$W_{\mathbb{R}P^3}(d, l) = \sum_{\substack{a+b=d \\ 0 \leq a < b}} (-1)^a (d - 2a) W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a, b), l).$$

By symmetry reasons, G. Mikhalkin proved that $W_{\mathbb{R}P^3}(d, l) = 0$ for even d (see also Remark 4.4). The invariants $W_{\mathbb{R}P^3}(d, d)$ have been computed in [GZ13].

We also establish a corresponding relation between the Gromov-Witten invariants of $\mathbb{C}P^3$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$. We denote by $GW_{\mathbb{C}P^3}(d)$ the count of degree d rational curves in $\mathbb{C}P^3$ passing through $2d$

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generic points and by $GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a, b)$ the count of rational curves of bidegree (a, b) in $\mathbb{C}P^1 \times \mathbb{C}P^1$ passing through $2(a + b) - 1$ generic points.

Theorem 2. *For every positive integer d, one has*

$$GW_{\mathbb{C}P^3}(d) = \sum_{\substack{a+b=d \\ 0 \le a < b}} (d - 2a)^2 GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a, b).$$

The idea of the proof begins with a result of J. Kollár, [Kol15, Proposition 3], which can be summarised as follows. A non-degenerate elliptic curve C_0 of degree 4 in $\mathbb{C}P^3$ generates a pencil of quadrics $\mathcal{Q}$ with base locus C_0. If the configuration $\underline{x}$ of $2d$ points is contained in C_0, every curve in degree d, passing through all points in $\underline{x}$, is contained in a non-singular quadric of $\mathcal{Q}$, where it is of bidegree (a, b), with $a \ne b$. Furthermore, all such quadrics can be recovered by a computation in the Jacobian of C_0 and in particular there are exactly $(d - 2a)^2$ of them. To complete the proof of Theorem 2 from [Kol15, Proposition 3], an additional transversality argument is needed: one has to show that a configuration $\underline{x}$ contained in C_0 can be chosen so that the number of the degree d curves passing through $\underline{x}$ is indeed the corresponding Gromov-Witten invariant of $\mathbb{C}P^3$. We prove that this is indeed the case in Section 4. The proof of Theorem 1 further requires a comparison of the signs of the curves which enter in the definitions of the Welschinger invariants of $\mathbb{C}P^3$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$. This is accomplished in Section 3.

Theorems 1 and 2 relate the enumerative geometry of $\mathbb{C}P^3$ on one hand and of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and of elliptic curves on the other hand. Both theorems can certainly be generalised to $\mathbb{C}P^3$ blown up in a small number of points. It would be interesting to understand further generalisations.

The paper is organised as follows. We start by recalling the definition of the Welschinger invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^3$ in Section 1. In Section 2 we review some standard facts concerning elliptic curves and pencils of quadrics in $\mathbb{C}P^3$. In Section 3 we investigate properties of rational curves in a smooth quadric of $\mathbb{C}P^3$ and establish the comparison of their two Welschinger signs. We combine results from Sections 2 and 3 in Section 4 to provide a transversality argument that allows us to deduce Theorem 2 from [Kol15, Proposition 3] and we prove Theorem 1. We end this paper with explicit computations of Welschinger invariants of $\mathbb{C}P^3$ together with further qualitative results and comments about them.

1. Welschinger invariants

1.1. **Welschinger invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1$.** Given $a, b \in \mathbb{Z}_{\ge 0}$ and $\underline{x}$ a generic real configuration of $2(a + b) - 1$ points in $\mathbb{C}P^1 \times \mathbb{C}P^1$, let $\mathbb{R}\mathcal{C}'(\underline{x})$ be the set of all real rational curves of bidegree (a, b) in $\mathbb{C}P^1 \times \mathbb{C}P^1$ passing through all points in $\underline{x}$. One can associate a sign $(-1)^{s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(C)}$ to each real curve C in $\mathbb{R}\mathcal{C}'(\underline{x})$ such that the sum

$$W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a, b), l) = \sum_{C \in \mathbb{R}\mathcal{C}'(\underline{x})} (-1)^{s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(C)}$$

only depends on a, b, and the number l of pairs of complex conjugated points in $\underline{x}$; see [Wel05a]. Since $\underline{x}$ is generic, every curve $C \in \mathbb{R}\mathcal{C}'(\underline{x})$ is nodal. A real node of C is either the intersection of two real branches of C, or the intersection of two complex conjugated branches. The node is called hyperbolic in the former and elliptic in the latter case. The number $s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(C)$ is defined to be

the number of elliptic real nodes of C.

The parity of $s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(C)$ can be interpreted in terms of a spin structure on $\mathbb{R}P^1 \times \mathbb{R}P^1$ as follows. Since a curve $C \in \mathbb{R}\mathcal{C}'(\underline{x})$ is nodal, it has a parametrisation $f : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ which is a real algebraic immersion. A choice of a trivialisation of $T\mathbb{R}P^1$ induces a trivialisation

$$\phi_0 : T(\mathbb{R}P^1 \times \mathbb{R}P^1) \to \mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}^2.$$

The canonical orientation and scalar product on $\mathbb{R}^2$ induce, via ϕ_0, an orientation and a Riemannian metric on $T(\mathbb{R}P^1 \times \mathbb{R}P^1)$. In their turn, the latter induce a trivialisation ϕ and a Riemannian metric on the $\mathbb{R}$-vector bundle $f^*_{|\mathbb{R}P^1} T(\mathbb{R}P^1 \times \mathbb{R}P^1)$. The tangent bundle $T\mathbb{R}P^1$ is naturally a subbundle of $f^*_{|\mathbb{R}P^1} T(\mathbb{R}P^1 \times \mathbb{R}P^1)$; let $\mathcal{N}$ be its orthogonal subbundle. Since $w_1(T\mathbb{R}P^1) = 0$, there exist a nowhere vanishing smooth section $\sigma_T : \mathbb{R}P^1 \to T\mathbb{R}P^1$. Let $\sigma_{\mathcal{N}} : \mathbb{R}P^1 \to \mathcal{N}$ be a section such that $(\sigma_T, \sigma_{\mathcal{N}})$ is a positive basis of $f^*_{|\mathbb{R}P^1} T(\mathbb{R}P^1 \times \mathbb{R}P^1)$ and N be the parity of the number of times the basis $\phi \circ (\sigma_T, \sigma_{\mathcal{N}})$ rotates around the canonical basis of $\mathbb{R}^2$ as one goes around $\mathbb{R}P^1$. In other words, the number N is the parity of the degree of the Gauß map of $f(\mathbb{R}P^1)$. Note that it does not depend on the choice of the trivialisation of $T\mathbb{R}P^1$.

Lemma 1.1. *One has $s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) = N \mod 2$.*
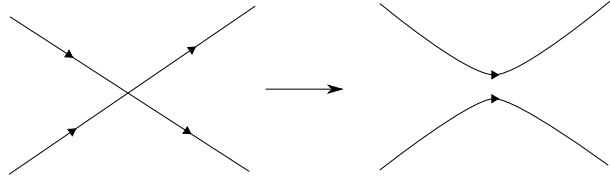

FIGURE 1. Smoothing of $f(\mathbb{R}P^1)$

Proof. We equip $f(\mathbb{R}P^1)$ with the orientation induced by σ_T. By smoothing each node of $f(\mathbb{R}P^1)$ as depicted in Figure 1, we obtain a collection γ of n disjoint oriented circles embedded in $\mathbb{R}P^1 \times \mathbb{R}P^1$. The sum of the Gauß index of all components of γ is the Gauß index of $f(\mathbb{R}P^1)$ and we have

$$n = 1 + \kappa \mod 2,$$

where κ is the number of hyperbolic nodes of $f(\mathbb{C}P^1)$. Every embedded circle realising the zero class in $H_1(\mathbb{R}P^1 \times \mathbb{R}P^1; \mathbb{Z})$, has Gauß index ± 1. Every embedded circle, which is non-trivial in the homology of $\mathbb{R}P^1 \times \mathbb{R}P^1$, is isotopic to a closed geodesic. In particular, it has Gauß index 0 and represents a class $(p, q) \in H_1(\mathbb{R}P^1 \times \mathbb{R}P^1; \mathbb{Z})$, with p and q relatively prime. Furthermore, since the components of the set γ do not intersect, all non-trivial components must represent (up to orientation) the same class $(p, q) \in H_1(\mathbb{R}P^1 \times \mathbb{R}P^1; \mathbb{Z})$ and the class of $f(\mathbb{R}P^1)$ can be written as (mp, mq) for some $m \in \mathbb{Z}$. Since the number N is equal to the parity of the number of homologically trivial components in γ, i.e. $n - m \mod 2$, we have

$$N = \kappa + 1 + m \mod 2.$$

If $f(\mathbb{C}P^1)$ has bidegree (a, b), by the adjunction formula it has exactly $(a-1)(b-1)$ nodes, κ of which are hyperbolic. Thus,

$$N = (a-1)(b-1) + 1 + m + s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) \mod 2.$$

Since $(mp, mq) = (a, b) \mod 2$, we have that m is even if and only if both a and b are even and so

$$s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) = N \mod 2,$$

as announced. $\qquad \square$

1.2. **Welschinger invariants of $\mathbb{C}P^3$.** Given $d \in \mathbb{Z}_{\geq 0}$ and $\underline{x}$ a generic real configuration of $2d$ points in $\mathbb{C}P^3$, let $\mathbb{R}\mathcal{C}(\underline{x})$ be the set of all real rational curves of degree d in $\mathbb{C}P^3$ passing through all points in $\underline{x}$. One can again associate a sign $(-1)^{s_{\mathbb{R}P^3}(C)}$ to each real curve C in $\mathbb{R}\mathcal{C}(\underline{x})$ such that the sum

$$W_{\mathbb{R}P^3}(d, l) = \sum_{C \in \mathbb{R}\mathcal{C}(\underline{x})} (-1)^{s_{\mathbb{R}P^3}(C)}$$

only depends on d and the number l of pairs of complex conjugated points in $\underline{x}$; see [Wel05b]. We now recall the definition of $s_{\mathbb{R}P^3}(C)$.

We fix once and for all an orientation on $\mathbb{R}P^3$. The projective space $\mathbb{R}P^3$ is spin and we may choose a trivialisation $\phi_0 : T\mathbb{R}P^3 \to \mathbb{R}P^3 \times \mathbb{R}^3$ of its tangent bundle compatible with the chosen orientation. The pullback by ϕ_0 of the canonical Euclidean scalar product on $\mathbb{R}^3$ provides a Riemannian metric on $\mathbb{R}P^3$. Let $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ be a real algebraic immersion. The trivialisation of $T\mathbb{R}P^3$ induces a trivialisation and a Riemannian metric on the $\mathbb{R}$-vector bundle $f^*_{|\mathbb{R}P^1} T\mathbb{R}P^3$. The tangent bundle $T\mathbb{R}P^1$ is naturally a $\mathbb{R}$-subbundle of $f^*_{|\mathbb{R}P^1} T\mathbb{R}P^3$; let $\mathcal{N}_{\mathbb{R}}$ be its orthogonal $\mathbb{R}$-subbundle. Choose an orientation of $\mathbb{R}P^1$ and a positive orthonormal section σ_T of $T\mathbb{R}P^1$. Given a line $\mathbb{R}$-subbundle E of $\mathcal{N}_{\mathbb{R}}$ with a non-vanishing smooth section σ_E, such that (σ_T, σ_E) is an orthonormal section of $T\mathbb{R}P^1 \oplus E$, there exists a unique choice of a section $\sigma_{\mathcal{N}}$ of $\mathcal{N}_{\mathbb{R}}$, such that $(\sigma_T, \sigma_E, \sigma_{\mathcal{N}})$ is a positive orthonormal section of $f^*_{|\mathbb{R}P^1} T\mathbb{R}P^3$. Combining with the trivialisation ϕ_0 of the latter bundle, this section defines a loop in $SO_3(\mathbb{R})$. We define

$$s(E) \in \{0, 1\}$$

to be the number realised by this loop in $\pi_1(SO_3(\mathbb{R})) = \{0, 1\}$. Note that $s(E)$ only depends on the isotopy class of E as an $\mathbb{R}$-subbundle of $\mathcal{N}_{\mathbb{R}}$ and on the homotopy class of the restriction of ϕ_0 to $T\mathbb{R}P^3_{|f(\mathbb{R}P^1)}$.

Let d be the degree of $f(\mathbb{C}P^1)$ in $\mathbb{C}P^3$. The quotient $\mathcal{N}_{\mathbb{C}}$ of $f^*T\mathbb{C}P^3$ by $T\mathbb{C}P^1$ is a holomorphic vector bundle over $\mathbb{C}P^1$ with first Chern class $4d - 2$. Suppose that the curve $f(\mathbb{C}P^1)$ is *balanced*, i.e. $\mathcal{N}_{\mathbb{C}}$ is isomorphic to the holomorphic bundle $\mathcal{O}_{\mathbb{C}P^1}(2d-1) \oplus \mathcal{O}_{\mathbb{C}P^1}(2d-1)$. In this case, there is a consistent way of choosing a line $\mathbb{R}$-subbundle $L(f)$ of the $\mathbb{R}$-bundle $\mathcal{N}_{\mathbb{R}}$ as follows. Holomorphic line subbundles of $\mathcal{N}_{\mathbb{C}}$ are in one-to-one correspondence with rational functions $F : \mathbb{C}P^1 \to \mathbb{C}P^1$: a fiber of such subbundle over the point u has equation $w = F(u)z$, where $(u, z, w) \subset \mathbb{C}^3$ are local coordinates on $\mathcal{N}_{\mathbb{C}}$ such that both (u, z) and (u, w) are local coordinates on $\mathcal{O}_{\mathbb{C}P^1}(2d-1)$. Moreover, a holomorphic line subbundle of $\mathcal{N}_{\mathbb{C}}$ defined by the rational function F has degree $2d - 1 - \deg F$ and is real if and only if F is real. In particular, up to real isotopy, there exists a unique holomorphic real line subbundle of $\mathcal{N}_{\mathbb{C}}$ of degree $2d - 1$ and two holomorphic real line subbundles of $\mathcal{N}_{\mathbb{C}}$ of degree $2d - 2$, depending on whether $F_{|\mathbb{R}P^1}$ is orientation preserving or not. Phrased differently, the two (real isotopy classes of) holomorphic real line subbundles of $\mathcal{N}_{\mathbb{C}}$ of degree $2d-2$ are characterised by the direction in which the real part of a fiber rotates in $\mathbb{R}\mathcal{N}_{\mathbb{C}}$. The Riemannian metric on $\mathbb{R}P^3$ identifies the $\mathbb{R}$-bundles $\mathcal{N}_{\mathbb{R}}$ and $\mathbb{R}\mathcal{N}_{\mathbb{C}}$. In particular, the section σ_T of $T\mathbb{R}P^1$ together with the orientation of $\mathbb{R}P^3$ induce an orientation on the bundle $\mathbb{R}\mathcal{N}_{\mathbb{C}}$: given $u \in \mathbb{R}P^1$, a positive basis of the fiber at u of $f^*_{|\mathbb{R}P^1} T\mathbb{R}P^3$ is formed by $(\sigma_T(u), v_1, v_2)$ with (v_1, v_2) a positive basis of $\mathbb{R}\mathcal{N}_{\mathbb{C}}$. We

denote by $L(f)$ (resp. $\overline{L}(f)$) the isotopy class of the real part of the degree $2d - 2$ line subbundle of $\mathcal{N}_\mathbb{C}$ whose real fibers rotate positively (resp. negatively) in local holomorphic coordinates on $\mathcal{N}_\mathbb{C}$ defining a real holomorphic splitting $\mathcal{N}_\mathbb{C} = \mathcal{O}_{\mathbb{C}P^1}(2d-1) \oplus \mathcal{O}_{\mathbb{C}P^1}(2d-1)$. Since $L(f)$ and $\overline{L}(f)$ differ by exactly one full rotation, which is a generator of $\pi_1(SO_2(\mathbb{R}))$, we have $s(L(f)) \neq s(\overline{L}(f))$.

For a generic real configuration $\underline{x}$ of points in $\mathbb{R}P^3$, every curve C in $\mathbb{R}\mathcal{C}(\underline{x})$ is parametrised by a balanced immersion $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ and the number $s_{\mathbb{R}P^3}(C)$ is defined as

$$s_{\mathbb{R}P^3}(C) = s(L(f)).$$

The number $s_{\mathbb{R}P^3}(C)$ is independent of the parametrisation but in general depends on the choice of a trivialisation ϕ_0. In the remaining of this note, we always assume that ϕ_0 is chosen so that $s_{\mathbb{R}P^3}(D) = 0$ for a line D in $\mathbb{C}P^3$. [1]

2. Elliptic curves and pencil of quadrics in $\mathbb{C}P^3$

In this section we recall some known facts about the Picard group of complex and real elliptic curves, their torsion points, and their relation with pencils of quadrics in $\mathbb{C}P^3$.

2.1. Torsion points of complex and real elliptic curves.
Let C_0 be a complex elliptic curve. Recall that a choice of $p_0 \in C_0$ induces an isomorphism

$$\begin{array}{rlc} \psi : & C_0 & \longrightarrow \quad \mathrm{Pic}_0(C_0) \\ & p & \longmapsto \quad [p] - [p_0], \end{array}$$

which induces in its turn a group structure on C_0. Geometrically, writing C_0 as the quotient of $\mathbb{C}$ by a full rank lattice Λ for which p_0 is the orbit of 0, the group structure induced by ψ on C_0 is simply the group structure inherited from $(\mathbb{C}, +)$ by the quotient map. This description allows to easily describe torsion points of order d on C_0: if $\Lambda = u\mathbb{Z} + v\mathbb{Z}$, with u and v two complex numbers, then the set of solutions of

$$(1) \qquad\qquad\qquad\qquad dp = 0$$

is a group of order d^2 isomorphic to $\mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ and generated by $\frac{1}{d}u$ and $\frac{1}{d}v$.

Analogously, the map ψ induces the following series of isomorphisms Ψ_d with $d \in \mathbb{Z}$:

$$\begin{array}{rlc} \Psi_d : & \mathrm{Pic}_d(C_0) & \longrightarrow \quad\quad C_0 \\ & \sum_{i=1}^d [p_i] & \longmapsto \quad \psi^{-1}\left(\sum_{i=1}^d p_i - dp_0\right) \end{array}$$

satisfying

$$\Psi_d(E) + \Psi_{d'}(E') = \Psi_{d+d'}(E + E').$$

Suppose now that C_0 is real, with $\mathbb{R}C_0 \neq \emptyset$, and that $p_0 \in \mathbb{R}C_0$. If $\mathbb{R}C_0$ is not connected, the connected component of $\mathbb{R}C_0$ containing p_0 is called the *pointed component* of $\mathbb{R}C_0$. The real structure on C_0 induces a real structure on $\mathrm{Pic}_d(C_0)$ for every $d \in \mathbb{Z}$ and the maps Ψ_d are all real maps (see [GH81]). Recall also (see [Nat90]) that C_0 can be expressed as $\mathbb{C}/\Lambda$ with the real structure inherited by the complex conjugation on $\mathbb{C}$, where Λ has one of the following forms:

- $\Lambda = u\mathbb{Z} + iv\mathbb{Z}$ with u and v two real numbers. In this case $\mathbb{R}C_0$ has two connected components: $\mathbb{R}/u\mathbb{Z}$ and $(\mathbb{R} + \frac{iv}{2})/u\mathbb{Z}$ (see Figure 2a). When d is even, both connected components of $\mathbb{R}C_0$ contain exactly d solutions of Equation (1). When d is odd, Equation (1) has exactly d real solutions, all located on the pointed component of $\mathbb{R}C_0$.

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[1]Such a choice is possible since every trivialization of $T\mathbb{R}P^3$ over the 2-skeleton extends to the 3-skeleton. The two homotopy classes of trivializations over the 2-skeleton correspond to different values of $s_{\mathbb{R}P^3}(D)$.

- $\Lambda = u\mathbb{Z} + \overline{u}\mathbb{Z}$ with u a complex number. In this case $\mathbb{R}C_0 = \mathbb{R}/(u + \overline{u})\mathbb{Z}$ is connected (see Figure 2b). Equation (1) has exactly d real solutions for any d.

a) A maximal real elliptic curve   b) A real elliptic curve with a connected real part
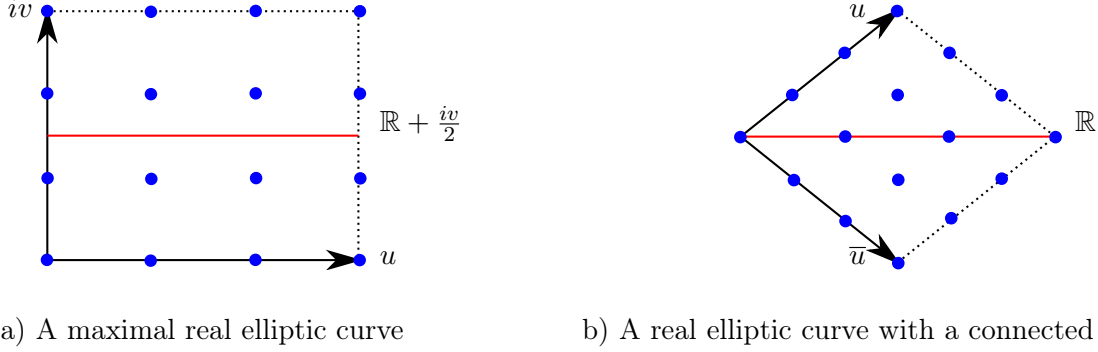
FIGURE 2. Uniformisation of real elliptic curves with a non-empty real part. The points represent the solutions of $3p = 0$.

In this paper, all considered elliptic curves are assumed to be equipped with a distinguished point p_0, which is real if the curve is real. In particular, we always identify $\mathrm{Pic}_d(C_0)$ with C_0 via the map Ψ_d.

2.2. Pencils of quadrics.

Let $\mathcal{Q}$ be a pencil of quadrics in $\mathbb{C}P^3$, i.e. a line in the space $\mathbb{C}P^9$ of quadrics in $\mathbb{C}P^3$. We assume that $\mathcal{Q}$ is generic enough so that the base locus of $\mathcal{Q}$ is a non-degenerate elliptic curve C_0 of degree 4 in $\mathbb{C}P^3$. Conversely, every non-degenerate elliptic curve C_0 of degree 4 in $\mathbb{C}P^3$ defines a pencil of quadrics with base locus C_0. We denote by $h \in C_0 \simeq \mathrm{Pic}_4(C_0)$ the hyperplane section class.

A non-singular quadric in $\mathbb{C}P^3$ is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ and admits a ruling by two families of lines in $\mathbb{C}P^3$; let D_1 and D_2 be two lines representing the families. Since C_0 is of bidegree $(2,2)$ in the non-singular quadrics of $\mathcal{Q}$, any such quadric defines two elements $E_i = D_i \cap C_0$ of $C_0 \simeq \mathrm{Pic}_2(C_0)$, whose sum is h. Conversely, given $E \in \mathrm{Pic}_2(C_0)$, the union of all lines in $\mathbb{C}P^3$, whose intersection with C_0 is a divisor defining E or $h - E$, is a quadric Q_E in $\mathcal{Q}$. Hence, the map

$$\pi_{\mathcal{Q}}: \quad C_0 \simeq \mathrm{Pic}_2(C_0) \quad \longrightarrow \quad \mathcal{Q}$$
$$E \quad \longmapsto \quad Q_E$$

is a ramified covering of degree two. The ramification values of $\pi_{\mathcal{Q}}$ correspond to singular quadrics in $\mathcal{Q}$, which are all quadratic cones. The corresponding critical points of $\pi_{\mathcal{Q}}$ are the solutions of the equation $2E = h$ and so any two of them differ by a torsion point of order 2. Thus, $\mathcal{Q}$ contains exactly 4 distinct singular quadrics, in accordance with the Riemann-Hurwitz formula.

Suppose now that C_0 is real with $\mathbb{R}C_0 \neq \emptyset$. In this case, the pencil $\mathcal{Q}$ is real and the fiber over a regular value $Q \in \mathbb{R}\mathcal{Q}$ of $\pi_{\mathcal{Q}}$ consists of two real points (resp. two complex conjugate points) if and only if $\mathbb{R}Q$ is homeomorphic to $S^1 \times S^1$ (resp. S^2). By the description of the real torsion points of C_0 given in Section 2.1, we have the following possibilities for the map $\pi_{\mathcal{Q}|\mathbb{R}C_0}$:

- $\mathbb{R}C_0$ is not connected and h is on the non-pointed component of $\mathbb{R}C_0$ (equivalently, both components of $\mathbb{R}C_0$ are non-trivial in $\pi_1(\mathbb{R}P^3)$): none of the four singular fibers of $\mathcal{Q}$ are real (see Figure 3a),
- $\mathbb{R}C_0$ is not connected and h is on the pointed component of $\mathbb{R}C_0$ (equivalently, both components of $\mathbb{R}C_0$ are trivial in $\pi_1(\mathbb{R}P^3)$): the four singular fibers of $\mathcal{Q}$ are real (see Figure 3b),

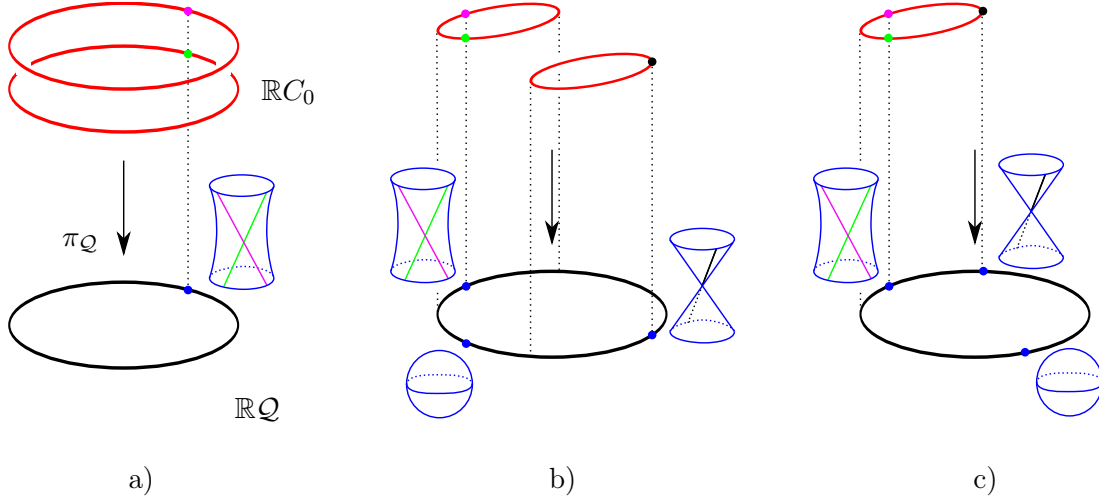- $\mathbb{R}C_0$ is connected: exactly two singular fibers of $\mathcal{Q}$ are real (see Figure 3c).

FIGURE 3. Real pencils of quadrics

3. RATIONAL CURVES ON A SMOOTH QUADRIC OF $\mathbb{C}P^3$

In this section we study properties of rational curves in a smooth quadric in $\mathbb{C}P^3$ and establish a comparison of the signs of a real curve as an element of the quadric and of the projective space.

Throughout the section we fix a smooth quadric Q of $\mathbb{C}P^3$. In particular, Q is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ and we denote by D_1 and D_2 two distinct intersecting lines in Q. Given an algebraic immersion $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ contained in Q, there is an exact sequence of holomorphic vector bundles over $\mathbb{C}P^1$:

$$(2) \qquad\qquad 0 \to \mathcal{N}'_{\mathbb{C}} \to \mathcal{N}_{\mathbb{C}} \to f^*\mathcal{N}_Q \to 0,$$

where $\mathcal{N}'_{\mathbb{C}}$ is the quotient bundle $f^*TQ/T\mathbb{C}P^1$, $\mathcal{N}_{\mathbb{C}}$ is the quotient bundle $f^*T\mathbb{C}P^3/T\mathbb{C}P^1$, and $\mathcal{N}_Q = T\mathbb{C}P^3|_Q/TQ$ is the normal bundle of Q in $\mathbb{C}P^3$. When Q and f are real, we also have the corresponding real bundles $\mathbb{R}\mathcal{N}'_{\mathbb{C}}, \mathbb{R}\mathcal{N}_{\mathbb{C}}$, and $\mathbb{R}\mathcal{N}_Q$ fitting in an exact sequence as above.

Proposition 3.1. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ be an algebraic immersion such that $f(\mathbb{C}P^1)$ is contained in Q, where it has bidegree (a, b), with $a \neq b$. Then, f is balanced.*

Proof. We want to prove that the normal bundle $\mathcal{N}_{\mathbb{C}}$ is isomorphic to the holomorphic bundle $\mathcal{O}_{\mathbb{C}P^1}(2d-1) \oplus \mathcal{O}_{\mathbb{C}P^1}(2d-1)$, where $d = a + b$. By the adjunction formula, the line bundle $\mathcal{N}'_{\mathbb{C}}$ has degree $2d - 2$ and thus $f^*\mathcal{N}_Q$ has degree $2d$. Hence, the map f is balanced if and only if the sequence (2) does not split. According to [GH83, Theorem 4.f.3], whose proof extends to immersions, this sequence splits if and only if $f(\mathbb{C}P^1)$ is a complete intersection. Since $a \neq b$, this is not the case.

We briefly recall the main lines of the proof of [GH83, Theorem 4.f.3]. Suppose that the above exact sequence splits and let $C_0 \subset Q$ be an elliptic curve of bidegree $(2, 2)$ intersecting $f(\mathbb{C}P^1)$ transversely. Then, there exists a holomorphic section σ of $\mathcal{N}_{\mathbb{C}}$, which vanishes only at the $2d$ points of $f(\mathbb{C}P^1) \cap C_0$. If $\mathcal{Q}$ denotes the pencil of quadrics in $\mathbb{C}P^3$ with base locus C_0, the section σ corresponds to a first order deformation $f_\varepsilon : \mathbb{C}P^1 \to Q_\varepsilon$ of f in the pencil $\mathcal{Q}$. Since σ vanishes at the $2d$ points in $f(\mathbb{C}P^1) \cap C_0$, the divisor class realised in $\mathrm{Pic}_{2d}(C_0)$ by $f_\varepsilon(\mathbb{C}P^1) \cap C_0$ is constant. On the other hand, as explained in Section 2.2, the quadric Q_ε in $\mathcal{Q}$ is determined by the class $E_{1,\varepsilon}$

realised by $D_{1,\varepsilon} \cap C_0$ in $\mathrm{Pic}_2(C_0)$, where $D_{i,\varepsilon}$ is the deformation in $\mathcal{Q}_\varepsilon$ of D_i. Since $f_\varepsilon(\mathbb{C}P^1)$ realises the class $aD_{1,\varepsilon} + bD_{2,\varepsilon} = (a - b)D_{1,\varepsilon} + bH$ in $\mathrm{Pic}(\mathbb{C}P^1 \times \mathbb{C}P^1)$, where H is the hyperplane section, $f_\varepsilon(\mathbb{C}P^1) \cap C_0$ realises the class $(a - b)E_{1,\varepsilon} + bh$ in $\mathrm{Pic}_{2d}(C_0)$. However, the class h is constant along the deformation, whereas the class $(a - b)E_{1,\varepsilon}$ is not, since $a - b \neq 0$. This is a contradiction. $\square$

Suppose now that Q is real with a real part homeomorphic to $\mathbb{R}P^1 \times \mathbb{R}P^1$ and that D_1 and D_2 are also real. Recall that, for a real algebraic immersion $f : \mathbb{C}P^1 \to \mathbb{C}P^3$, the isotopy classes $L(f)$ and $\overline{L}(f)$ and the numbers $s(E)$, for an orientable $\mathbb{R}$-subbundle $E \subset \mathbb{R}\mathcal{N}_\mathbb{C}$, have been defined in Section 1.2.

Proposition 3.2. *Let $\mathbb{R}\mathcal{N}'_i$ be the real part of the normal bundle of D_i in Q, considered as a subbundle of the real part of its normal bundle in $\mathbb{C}P^3$. Then, one has $s(\mathbb{R}\mathcal{N}'_1) \neq s(\mathbb{R}\mathcal{N}'_2)$.*

Proof. Given a line D in $\mathbb{C}P^3$, the holomorphic line subbundles of degree 1 of its normal bundle in $\mathbb{C}P^3$ correspond precisely to the planes of $\mathbb{C}P^3$ containing D. In particular, the normal bundle $\mathbb{R}\mathcal{N}'_i$ realises the isotopy class $L(D_i)$ if and only if it rotates in $\mathbb{R}P^3$ around $\mathbb{R}D_i$ in the positive direction. Since $\mathbb{R}\mathcal{N}'_1$ and $\mathbb{R}\mathcal{N}'_2$ do not rotate in the same direction, the result follows. $\square$

We will say that (D_1, D_2) is *the positive basis* of $H_2(Q; \mathbb{Z})$ if $s(\mathbb{R}\mathcal{N}'_1) = 0$.

Corollary 3.3. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ be a real algebraic immersion such that $f(\mathbb{C}P^1)$ is contained in Q, where it has a bidegree (a, b) in the positive basis, with $a \neq b$. Then, one has*

$$s(\mathbb{R}\mathcal{N}'_\mathbb{C}) = s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) + b.$$

Proof. The proof is similar to the proof of Lemma 1.1. Equip $\mathbb{R}P^1$ with some orientation and smooth each node of $f(\mathbb{R}P^1)$, as depicted in Figure 1, in order to obtain a collection γ of oriented circles embedded in $\mathbb{R}P^1 \times \mathbb{R}P^1$. The loop in $\pi_1(SO_3(\mathbb{R}))$ defined by $\mathbb{R}\mathcal{N}'_\mathbb{C}$ is freely homotopic to the product of the loops in $\pi_1(SO_3(\mathbb{R}))$ defined by $T\mathbb{R}Q_{|\gamma_i}/T\gamma_i$ for γ_i ranging over elements of γ. Any loop in γ realising the 0 class (resp. the class $p[\mathbb{R}D_1] + q[\mathbb{R}D_2]$ with $\gcd(p, q) = 1$) in $H_1(\mathbb{R}P^1 \times \mathbb{R}P^1; \mathbb{Z})$ defines a non-trivial loop (resp. q times the non-trivial loop) in $\pi_1(SO_3(\mathbb{R}))$. Now the end of the proof is similar to the end of the proof of Lemma 1.1. $\square$

Proposition 3.4. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ be a real algebraic immersion such that $f(\mathbb{C}P^1)$ is contained in Q, where it has a bidegree (a, b) in the positive basis, with $a \neq b$. Then, the holomorphic real line subbundle $\mathbb{R}\mathcal{N}'_\mathbb{C}$ of $\mathbb{R}\mathcal{N}_\mathbb{C}$ realises the real isotopy class $L(f)$ if and only if $a > b$.*

Proof. By convention, we have $s_{\mathbb{R}P^3}(D) = 0$ for a real line D in $\mathbb{C}P^3$. Hence, according to Proposition 3.2, the proposition is true for $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$.

Recall how the two isotopy classes $L(f)$ and $\overline{L}(f)$ are characterised: in local holomorphic coordinates on $\mathcal{N}_\mathbb{C}$ defining a real splitting $\mathcal{N}_\mathbb{C} = \mathcal{O}_{\mathbb{C}P^1}(2d - 1) \oplus \mathcal{O}_{\mathbb{C}P^1}(2d - 1)$, the classes $L(f)$ and $\overline{L}(f)$ rotate in different directions in $\mathbb{R}\mathcal{N}_\mathbb{C}$. The positive direction of rotation is determined by an orientation of $\mathbb{R}P^1$ and of $\mathbb{R}P^3$; the class $L(f)$ is by definition the one which rotates in the positive direction. Hence to prove the proposition, it is enough to find a point $u_0 \in \mathbb{R}P^1$ and a real holomorphic subbundle E of $\mathcal{N}_\mathbb{C}$ of degree $2d - 1$ such that

- the fibers of E and $\mathcal{N}'_\mathbb{C}$ over u_0 coincide and
- we can determine the mutual position of the real parts of the fibers of these bundles over a point $u \in \mathbb{R}P^1$ in a neighbourhood of u_0.

Let $C_0 \subset Q$ be a real elliptic curve of bidegree $(2, 2)$, with $\mathbb{R}C_0 \neq 0$, that intersects $f(\mathbb{C}P^1)$ transversely at some point $p_0 = f(u_0)$ with $u_0 \in \mathbb{R}P^1$. We denote by $\mathcal{Q}$ the real pencil of quadrics defined by C_0. In particular, $\mathcal{Q}$ is the realisation of a first order real deformation Q_ε of the real

quadric Q in $\mathbb{C}P^3$. Denoting by $D_{i,\varepsilon}$ the deformation of D_i in Q_ε and by $E_{i,\varepsilon} \in \mathrm{Pic}_2(C_0)$ the class realised by $D_{i,\varepsilon} \cap C_0$, recall that

$$(3) \qquad \frac{dE_{i,\varepsilon}}{d\varepsilon}|_{\varepsilon=0} \neq 0.$$

Let f_ε be a first order real deformation of f in the pencil $\mathcal{Q}$ such that $f_\varepsilon(\mathbb{C}P^1)$ passes through the $2d - 1$ points of $f(\mathbb{C}P^1) \cap C_0 \setminus \{p_0\}$ for all ε. This deformation corresponds to a non-null real holomorphic section $\sigma : \mathbb{C}P^1 \to \mathcal{N}_\mathbb{C}$ that vanishes on $f^{-1}(C_0 \setminus \{p_0\})$. Recall that, since $a \neq b$, the class realised by $f_\varepsilon(\mathbb{C}P^1) \cap C_0$ in $\mathrm{Pic}_{2d}(C_0)$ is not constant. In particular, $\sigma(u_0) \neq 0$. Let E be the real holomorphic subbundle of $\mathcal{N}_\mathbb{C}$ of degree $2d-1$ whose fiber over u_0 is the line generated by $\sigma(u_0)$.

Claim 1: σ is a section of E. Indeed, σ induces a holomorphic section of the bundle $\mathcal{N}_\mathbb{C}/E$ that vanishes at the $2d$ points of $f^{-1}(C_0)$. Since the latter bundle has degree $2d - 1$, this induced section must be the null section i.e. $\sigma(u) \in E$ for all $u \in \mathbb{C}P^1$.

Claim 2: the fibers of E and $\mathcal{N}'_\mathbb{C}$ over u_0 coincide. Indeed, the vector $\sigma(u_0)$ corresponds to the preimage of the deformation of p_0 in $f_\varepsilon(\mathbb{C}P^1) \cap C_0$. Since this deformation stays in C_0 by definition and C_0 sits in Q, we must have $\sigma(u_0) \in \mathcal{N}'_\mathbb{C}$.

Claim 3: the isotopy class realised by $\mathbb{R}\mathcal{N}'_\mathbb{C}$ is determined by the direction of the vector $\sigma(u_0)$. Let us denote by σ_Q the holomorphic section of $T\mathbb{C}P^3/TQ$ corresponding to the deformation $\mathcal{Q}$. Since both $\mathbb{R}P^3$ and $\mathbb{R}Q$ are orientable manifolds, we may also fix a smooth nowhere vanishing section λ_Q of the $\mathbb{R}$-vector bundle $T\mathbb{R}P^3/T\mathbb{R}Q$ (see Figure 4a for a local picture at p_0). The section σ_Q vanishes along C_0 and we denote by $\mathbb{R}Q_+$ the region of $\mathbb{R}Q \setminus \mathbb{R}C_0$, where σ_Q and λ_Q have the same direction (see Figure 4b). The choices of λ_Q and σ_Q induce an orientation on the real part of our source curve: we orient $\mathbb{R}P^1$ so that $f(\mathbb{R}P^1)$ points toward $\mathbb{R}Q_+$ at $f(u_0)$. The choice of a Riemannian metric on $\mathbb{R}P^3$ identifies the $\mathbb{R}$-bundle $f^*(T\mathbb{R}P^3/T\mathbb{R}Q)$ with the orthogonal of $\mathbb{R}\mathcal{N}'_\mathbb{C}$ in $\mathbb{R}\mathcal{N}_\mathbb{C}$. With this identification, the vector $\sigma(u)$ for $u \in \mathbb{R}P^1$ close enough to u_0 decomposes as

$$\sigma(u) = g_1(u)\sigma(u_0) + g_2(u)\lambda_Q(u),$$

where g_1 is a smooth function with $g_1(u_0) = 1$ and g_2 is a smooth function vanishing at u_0 and positive for $u > u_0$ (the local ordering of $\mathbb{R}P^1$ at u_0 is given by the orientation of $\mathbb{R}P^1$). In other words, the choice of λ_Q and σ_Q determine an orientation of the fiber of $\mathbb{R}\mathcal{N}_\mathbb{C}$ over u_0 together with a half-plane $\Pi \subset \mathbb{R}\mathcal{N}_{\mathbb{C}|u_0} \setminus \mathbb{R}\mathcal{N}'_{\mathbb{C}|u_0}$ containing $\sigma(u)$ when $u > u_0$ (see Figure 4 c, d). Now clearly, the direction in which $\mathbb{R}\mathcal{N}'_\mathbb{C}$ rotates with respect to $\mathbb{R}E$ depends only on the direction of $\sigma(u_0)$.

As in the proof of Proposition 3.1, the deformation p_ε of p_0 as an intersection point of $f_\varepsilon(\mathbb{C}P^1)$ and C_0 is determined by the condition that $f_\varepsilon(\mathbb{C}P^1) \cap C_0$ has to realise the class $(a - b)E_{1,\varepsilon} + bh$ in $\mathrm{Pic}_{2d}(C_0)$. Since the class h is constant, Inequality (3) implies that the direction of the vector

$$\frac{dp_\varepsilon}{d\varepsilon}|_{\varepsilon=0}$$

is the same as for the line D_1 if $a > b$ and opposite if $a < b$. Since the statement of the proposition holds for the classes $(1, 0)$ and $(0, 1)$, the proof is complete. □

Corollary 3.5. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ be a real algebraic immersion such that $f(\mathbb{C}P^1)$ is contained in Q, where it has a bidegree (a, b) in the positive basis, with $a \neq b$. Then, one has*

$$s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = \begin{cases} s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) + b + 1, & \text{if } a < b, \\ \\ s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) + b, & \text{if } a > b. \end{cases}$$
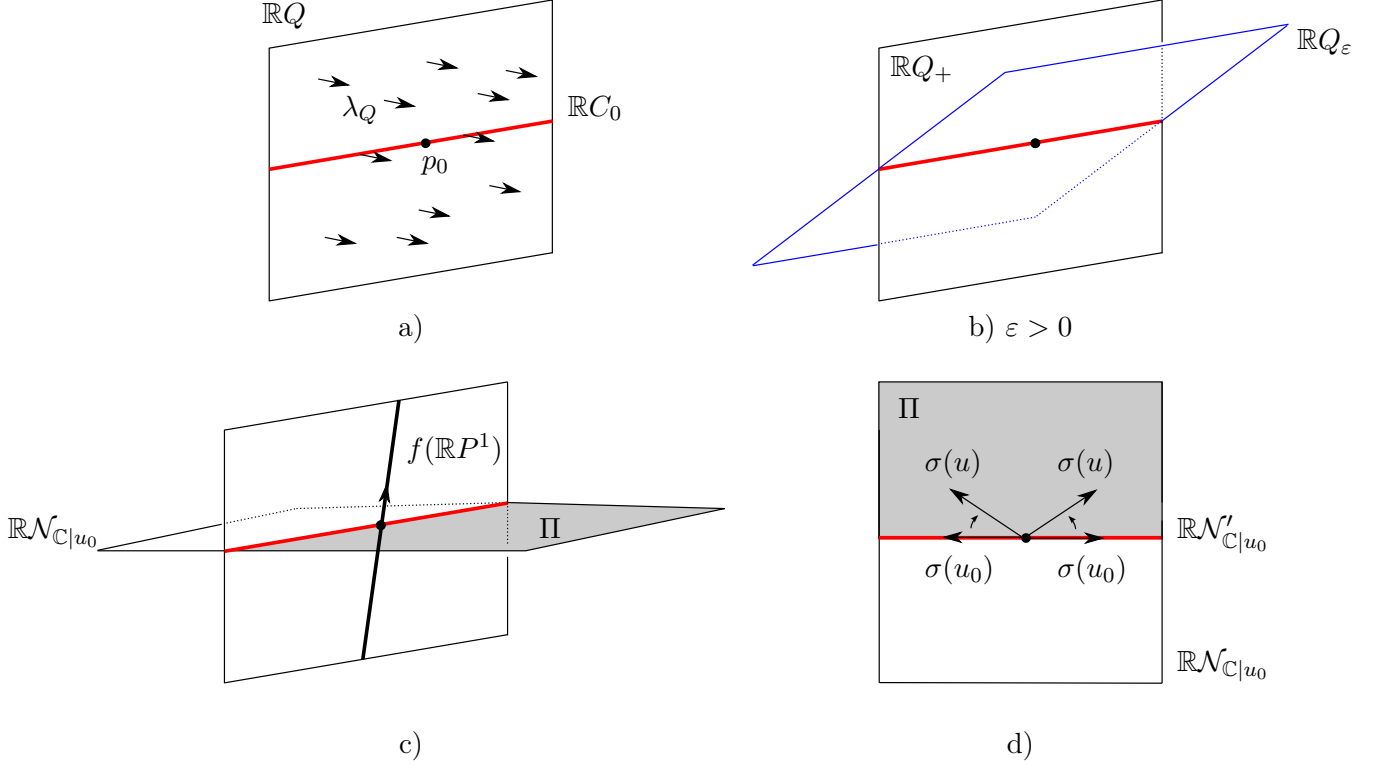
FIGURE 4. Determining the isotopy class of $\mathbb{R}\mathcal{N}'_{\mathbb{C}}$

4. PROOFS OF THEOREMS 1 AND 2

In this section we recall in Proposition 4.1 the main statement for our purposes from [Kol15] and we prove transversality results needed to deduce Theorem 2 from Proposition 4.1. We finish the section with the proof of Theorem 1.

Let C_0 be a non-degenerate elliptic curve of degree 4 in $\mathbb{C}P^3$ and $\underline{x} \subset C_0$ be a configuration of $2d$ distinct points. We denote by $\mathcal{Q}$ the pencil of quadrics induced by C_0 and by $\overline{\mathcal{C}}(\underline{x})$ the set of connected algebraic curves of arithmetic genus 0 and degree d in $\mathbb{C}P^3$ that contain $\underline{x}$. Recall that $h \in \mathrm{Pic}_4(C_0)$ denotes the hyperplane section.

Proposition 4.1 ([Kol15, Proposition 3])**.** *Suppose that the points in $\underline{x}$ are in general position in C_0. Then, every curve in $\overline{\mathcal{C}}(\underline{x})$ is irreducible and contained in a quadric of $\mathcal{Q}$. Furthermore, the quadrics of $\mathcal{Q}$, that contain a curve in $\overline{\mathcal{C}}(\underline{x})$, are exactly the images under $\pi_{\mathcal{Q}}$ of the solutions $E \in Pic_2(C_0)$ of the equation*

$$(4) \qquad\qquad (d - 2a)E = (d - a)h - \underline{x},$$

with $0 \le a < \frac{d}{2}$.

If Q is such a quadric and C is a curve in $\overline{\mathcal{C}}(\underline{x})$, then C is linearly equivalent in Q to $aD_1 + (d-a)D_2$, where D_1 (resp. D_2) is a line in Q whose intersection with C_0 is E (resp. $h - E$). Conversely, any irreducible rational curve C in Q, linearly equivalent to $aD_1 + (d - a)D_2$ and containing $2d - 1$ of the points in $\underline{x}$, is in $\overline{\mathcal{C}}(\underline{x})$ (i.e. contains $\underline{x}$).

Any two solutions of Equation (4) differ by a torsion point of order $d - 2a$; in particular, Equation (4) has exactly $(d - 2a)^2$ solutions in C_0. Let $\mathcal{M}^*_{0,2d}(\mathbb{C}P^3, d)$ be the space of stable maps $f : (\mathbb{C}P^1, x_1, \ldots, x_{2d}) \to \mathbb{C}P^3$ from $\mathbb{C}P^1$ with $2d$ marked points $x_1, \ldots, x_{2d}$, considered up to reparametrisation, whose image has degree d. The evaluation map ev is defined as

$$ev : \begin{array}{ccc} \mathcal{M}^*_{0,2d}(\mathbb{C}P^3, d) & \longrightarrow & (\mathbb{C}P^3)^{2d} \\ f & \longmapsto & (f(x_1), \ldots, f(x_{2d})). \end{array}$$

To complete the proof of Theorem 2, it remains to prove that one can choose $\underline{x} \subset C_0$ so that the map ev is regular at every curve in $\overline{\mathcal{C}}(\underline{x})$ and that each quadric of $\mathcal{Q}$, solution to Equation (4), contains exactly $GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a, d - a)$ elements of $\overline{\mathcal{C}}(\underline{x})$. This is done in the next two propositions. Denote by

$$V_n \subset C_0^n \subset (\mathbb{C}P^3)^n$$

the set of configurations of n distinct points on C_0.

Proposition 4.2. *Let Q be a quadric in $\mathcal{Q}$. Given an integer $a \in \{0, \ldots, d\}$ and $\underline{y}$ in V_{2d-1}, we denote by $\overline{\mathcal{C}}'_{Q,a}(\underline{y})$ the set of stable maps $f : (C, x_1, \ldots, x_{2d-1}) \to Q$, with C a connected nodal curve of arithmetic genus 0, such that $f(C)$ has bidegree $(a, d - a)$ and $f(\{x_1, \ldots, x_{2d-1}\}) = \underline{y} \in V_{2d-1}$. Then, there exists a dense open subset $U_{2d-1} \subset V_{2d-1}$ such that for every choice of a and $\underline{y} \in U_{2d-1}$, and for every stable map $f : (C, x_1, \ldots, x_{2d-1}) \to Q$ in $\overline{\mathcal{C}}'_{Q,a}(\underline{y})$, one has*

- *C is non-singular,*
- *f is an immersion.*

In particular, for every configuration $\underline{y} \in U_{2d-1}$, the quadric Q contains exactly $GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a, d-a)$ rational curves of bidegree $(a, d - a)$ and passing through $\underline{y}$.

Proof. Let $\mathcal{V}_a$ be the set of irreducible nodal rational curves in Q of bidegree $(a, d - a)$. This is a quasiprojective subvariety of dimension $2d - 1$ of the linear system $|aD_1 + (d - a)D_2|$ and we denote by $\overline{\mathcal{V}}_a$ its Zariski closure. We also denote by $\mathcal{U}_a$ the linear system on C_0 defined by the restriction to C_0 of the divisor $aD_1 + (d - a)D_2$. Since C_0 has bidegree $(2, 2)$ in Q, the linear system $\mathcal{U}_a$ has degree $2d$. By the Riemann-Roch theorem, every element of $\mathcal{U}_a$ is determined by $2d - 1$ of its points in C_0. In particular, every element $\underline{y}$ of V_{2d-1} induces an element $[\underline{y}]$ of $\mathcal{U}_a$. Since C_0 cannot be a component of a curve in $\overline{\mathcal{V}}_a$, the map

$$\phi : \begin{array}{ccc} \overline{\mathcal{V}}_a & \longrightarrow & \mathcal{U}_a \\ C_1 & \longmapsto & C_1 \cap C_0 \end{array}$$

is well defined, generically finite, and $\dim \phi(\overline{\mathcal{V}}_a \setminus \mathcal{V}_a) \le 2d - 2$. Hence if $\underline{y} \in V_{2d-1}$ is so that $[\underline{y}] \notin \phi(\overline{\mathcal{V}}_a \setminus \mathcal{V}_a)$, then for every element $f : (C, x_1, \ldots, x_{2d-1}) \to Q$ of $\overline{\mathcal{C}}'_{Q,a}(\underline{y})$, the curve $f(C)$ must be a nodal irreducible rational curve. In other words, we have $C = \mathbb{C}P^1$ and f is an immersion.

Let $\mathcal{M}^*_{0,2d-1}(Q, (a, d - a))$ be the space of stable maps $f : (\mathbb{C}P^1, x_1, \ldots, x_{2d-1}) \to Q$ from $\mathbb{C}P^1$ with $2d - 1$ marked points, considered up to reparametrisation, whose image has bidegree $(a, d - a)$. Since $Q \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$ is convex, the proof of [Wel05b, Lemma 1.3] implies that a point f in $\mathcal{M}^*_{0,2d-1}(Q, (a, d-a))$ is regular for the corresponding evaluation map if and only if it is an immersion. This completes the proof. $\square$

Proposition 4.3. *Regular values of ev contained in V_{2d} form a dense open subset $U_{2d} \subset V_{2d}$. In particular, if $\underline{x} \in U_{2d}$, then the set $\overline{\mathcal{C}}(\underline{x})$ contains exactly $GW_{\mathbb{C}P^3}(d)$ elements.*

Proof. By [Wel05b, Lemma 1.3], an element of $\mathcal{M}^*_{0,2d}(\mathbb{C}P^3, d)$ is a regular point of ev if and only if it is a balanced immersion. Let $\underline{x}$ be a configuration of $2d$ points on C_0 for which the conclusions of Proposition 4.1 hold. Note that the quadrics, solutions to Equation (4), do not change if $\underline{x}$ is

replaced by a configuration of points linearly equivalent to $\underline{x}$ in C_0. By the Riemann-Roch theorem, every effective divisor linearly equivalent to $\underline{x}$ in C_0 is determined by $2d-1$ points in C_0. Hence, according to Proposition 4.2, if $\underline{x}$ is chosen generically among all configurations of $2d$ points in C_0 linearly equivalent to a given generic divisor of degree $2d$, all elements in $\overline{\mathcal{C}}(\underline{x})$ will be immersions. Thus, we are left to show that every element $f : (\mathbb{C}P^1, x_1, \ldots, x_{2d}) \to \mathbb{C}P^3$ of $\overline{\mathcal{C}}(\underline{x})$ is balanced, which follows from Proposition 3.1. □

Proof of Theorem 2. Let U_{2d} be as in Proposition 4.3. Choose $\underline{x} \in U_{2d}$ and $\underline{y} \subset \underline{x}$ a set of $2d-1$ points in $\underline{x}$. We denote by $\mathcal{W}_a$ the set of quadrics in $\mathcal{Q}$ corresponding to a solution of Equation (4). As explained above, this set contains $(d-2a)^2$ elements. By Proposition 4.1, we know that the equality

$$|\overline{\mathcal{C}}(\underline{x})| = \sum_{0 \le a < b} \sum_{Q \in \mathcal{W}_a} |\overline{\mathcal{C}}'_{Q,a}(\underline{y})|$$

holds. By Proposition 4.3, we have $|\overline{\mathcal{C}}(\underline{x})| = GW_{\mathbb{C}P^3}(d)$. Since $\underline{x}$ is a regular value of the map ev, every map in $\overline{\mathcal{C}}(\underline{x})$ is an immersion $f : \mathbb{C}P^1 \to \mathbb{C}P^3$. Hence by Proposition 4.2, we have $|\overline{\mathcal{C}}'_{Q,a}(\underline{y})| = GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a, b)$. □

Proof of Theorem 1. Suppose now that C_0 is real, with $\mathbb{R}C_0 \ne 0$, and $\underline{x} \in U_{2d}$ is a real configuration of $2d$ points, containing at least one real point.

Let d be an odd positive integer and Q be a quadric containing a real map $f : \mathbb{C}P^1 \to \mathbb{C}P^3 \in \overline{\mathcal{C}}(\underline{x})$. According to Proposition 4.1, there exists an integer $a \in \{0, \ldots, \frac{d-1}{2}\}$ such that $f(\mathbb{C}P^1)$ has bidegree either $(a, d-a)$ or $(d-a, a)$ in the positive basis of Q. From Corollary 3.5, we deduce that

$$s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = \begin{cases} s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)), & \text{if } a \text{ is even,} \\[2mm] s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) + 1, & \text{if } a \text{ is odd.} \end{cases}$$

Thus, the total contribution to $W_{\mathbb{R}P^3}(d, l)$ of elements of $\overline{\mathcal{C}}(\underline{x})$, whose image is contained in Q, is $W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a, d-a), l)$ if a is even and $-W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a, d-a), l)$ if a odd. Equation (4) always has a real solution and two real solutions differ by a real torsion element of order $d-2a$. Hence, Equation (4) has exactly $d-2a$ real solutions for every $a \in \{0, \ldots, \frac{d-1}{2}\}$. Now the end of the proof is similar to the proof of Theorem 2. □

Remark 4.4. One can also prove the vanishing of $W_{\mathbb{R}P^3}(d, l)$ for d even and $l \le d-1$ in this way. This is also how J. Kollár exhibited configurations $\underline{x}$ for which $\overline{\mathcal{C}}(\underline{x})$ contains no real curves: if h is on the pointed component of $\mathbb{R}C_0$ and $\underline{x}$ is on the non-pointed component, then Equation (4) has no real solution.

5. Computations and further comments

We provide in Tables 1 and 2 the values of $W_{\mathbb{R}P^3}(d, l)$ for small values of d. The values $W_{\mathbb{R}P^3}(d, d)$ are taken from [GZ13]. Our values of $W_{\mathbb{R}P^3}(d, 0)$ agree with the ones computed in [BM07] and [ABLdM11].

The tables show that the values of the Welschinger invariants with 2 or 0 real points are the same in the computed degrees. This lead us to make the following conjecture.

Conjecture 5.1. *For every positive integer d, one has*

$$W_{\mathbb{R}P^3}(d, d-1) = W_{\mathbb{R}P^3}(d, d).$$

From the first values of $W_{\mathbb{R}P^3}(d, l)$, it might be tempting to conjecture that

$$(-1)^k W_{\mathbb{R}P^3}(2k+1, l) \ge (-1)^k W_{\mathbb{R}P^3}(2k+1, l+1) \ge 0.$$

d \ l	1	3	5	7	9	11	13
0	1	−1	45	−14589	17756793	−58445425017	426876362998821
1	1	−1	29	−6957	6717465	−18318948633	114201657733941
2		−1	17	−3093	2407365	−5495423913	29447853240537
3		−1	9	−1269	812157	−1571343273	7298043143697
4			5	−477	256065	−426170217	1732456594269
5			5	−173	75281	−109136649	392521356477
6				−85	21165	−26389305	84651531633
7				−85	6165	−6109369	17390628729
8					1993	−1401241	3432362709
9					1993	−336441	663105669
10						−136457	129344841
11						−136457	27607073
12							3991693
13							3991693

TABLE 1.

d \ l	15	17
0	−6061743911446054965	15224462564872441783409
1	−1414422922125979269	3149720751948303516897
2	−319737783634469757	633751084789301814083
3	−69876860779936989	123819546024578639718
4	−14727767907263157	23446928218635352817
5	−2985647746084965	4294618837478186313
6	−580664589588189	759270779118364253
7	−108170761670685	129334357769713247
8	−19320554509557	21207130905294257
9	−3327374698245	33506171960522913
10	−558961586685	5121214631258589
11	−93320976413	763120829396277
12	−16000904949	112222758491433
13	−2937725541	16596074817721
14	−1580831965	2542297019941
15	−1580831965	447392666733
16		−129358296175
17		−129358296175

TABLE 2.

However, both inequalities turn out not to hold in general, starting in degrees 17 and 19:

$$W_{\mathbb{R}P^3}(17,16) < 0 \quad \text{and} \quad -W_{\mathbb{R}P^3}(19,17) = 74131154312945 < 106335656443537 = -W_{\mathbb{R}P^3}(19,18).$$

In fact, the sign of $W_{\mathbb{R}P^3}$ seems to obey an analogous rule as the one observed in [Bru15] for the Welschinger invariants of $\mathbb{R}P^2$: as l goes from 0 to $d-1$, the numbers $(-1)^k W_{\mathbb{R}P^3}(2k+1, l)$ are first

positive and then, starting from some mysterious threshold, have an alternating sign. It would be interesting to investigate what governs the sign of the Welschinger invariants.

We end this paper by establishing a congruence modulo 4 between the Welschinger and the Gromov-Witten invariants, generalising results from [BM07] and [GZ13].

Proposition 5.2. *For every positive integers a, b, and $l \in \{0, \ldots, a+b-1\}$, one has*
$$GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a,b) = W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a,b),l) \mod 4.$$

Proof. Both invariants can be computed via the enumeration of marked floor diagrams (see [BM08]) with Newton polygon the rectangle with vertices
$$(0,0), \ (a,0), \ (0,b), \ \text{and} \ (a,b).$$

Any floor diagram with this Newton polygon has only floors with divergence 0. Hence, every marked floor diagram contributing to $GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a,b)$ has only non-negative real multiplicities. Now the result follows from the fact that the absolute value of a real multiplicity of a marked floor diagram is always equal modulo 4 to its complex multiplicity. □

Corollary 5.3. *For every positive integers d and $l \in \{0, \ldots, d\}$, one has*
$$GW_{\mathbb{C}P^3}(d) = (-1)^{\frac{(d-1)(d-2)}{2}} W_{\mathbb{R}P^3}(d,l) \mod 4.$$

Proof. When d is even, this follows from Theorem 2 and the vanishing of $W_{\mathbb{R}P^3}(d,l)$. When d is odd, this follows from Theorem 1, Proposition 5.2, and the congruence $(-1)^u = 2u + 1 \mod 4$. □

In [BM07] it was also shown that the inequality $(-1)^{\frac{(d-1)(d-2)}{2}} W_{\mathbb{R}P^3}(d,0) \geq 0$ holds for all d. It is not clear how to deduce this inequality from Theorem 1.

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